CROSS CURVATURE FLOW ON LOCALLY HOMOGENOUS THREE-MANIFOLDS (I)

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Abstract. Chow and Hamilton introduced the cross curvature flow on closed 3-manifolds with negative or positive sectional curvature. In this paper, we study the negative cross curvature flow in the case of locally homogenous metrics on 3-manifolds. In each case, we describe the long time behavior of the solutions of the corresponding ODE system.

1. Introduction

1.1. **Homogeneous evolution equations.** Hamilton's Ricci flow ([Hamilton 1982]) is the seminal and most successful example of the idea of deforming a Riemannian structure using a geometric evolution equation. Special cases arise when the metric is invariant under a group of transformations and this property is preserved by the flow. In particular, if the group of isometries of the original Riemannian structure is transitive, then the geometric evolution equation reduces to an ODE in the tangent space of an arbitrary fixed origin. In this spirit, the Ricci flow on locally homogeneous 3-manifolds was analyzed by Isenberg and Jackson [1992] , quasi-convergence of model geometries under the Ricci flow was studied by Knopf and McLeod [2001] , and the case of the Ricci flow on locally homogenous closed 4-manifolds was analyzed by Isenberg, Jackson and Lu [2006] . Lott [2007] interprets these results using the notion of groupoids and solitons.

This paper studies the asymptotic behavior of the (negative) cross curvature flow on locally homogeneous metrics in dimension 3. This flow was introduced by Chow and Hamilton [2004] and is (so far) specific to dimension 3. It depends on a sign choice (see Section 1.3 below). Chow and Hamilton conjectured that for any compact 3-manifold that admits a metric with negative sectional curvature, the (positive) cross curvature flow exists for all time and converge to a hyperbolic metric. Because of the structure of cross curvature flow equation, no general existence results are expected when sectional curvatures take different signs, which is the case for most homogeneous geometries. However, in the case of homogeneous geometries, there is no difficulties in defining, say, the negative curvature flow since the equations reduce to a system of ODEs. The choice of a sign mentioned above can then be interpreted simply as

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running the flow either forward of backward (although one should observe that it is not clear, a priori, which direction should be considered the forward direction).

These remarks prompted us to study the asymptotic forward *and* backward behaviors of the maximal two sided solutions of the cross curvature flow on homogeneous 3-manifolds. The present paper deals with the (forward) behavior of the negative cross curvature flow. The companion paper [Cao and Saloff-Coste 2007] deals with the backward behavior of the negative cross curvature flow (i.e., the forward behavior of the positive cross curvature flow). The backward behavior of the Ricci flow will be considered elsewhere. One interesting discovery is that, generically, the backward behavior of these flows is described by convergence (of the distance functions) to non-degenerate sub-Riemannian geometries. See [Cao and Saloff-Coste 2007]. Concerning the forward direction studied in this paper, it is interesting to compare the behavior of the Ricci flow to that of the negative cross curvature flow. Let us briefly describe the similarities and differences. In the case of geometries modeled on SU(2) and the Heisenberg group, the behavior of the Ricci flow and cross curvature flow are similar. On SU(2), both flows (in their normalized version) have solutions that exists for all positive time and converge towards round metrics. On closed manifolds with Heisenberg type geometries both normalized flows exist for all positive time and as t tends to infinity, they produce almost flat metrics. On closed manifolds of type $E(2)$, both normalized flows exists for all positive time. The normalized Ricci flow converges to a flat metric whereas the normalized negative cross curvature flow produces almost flat metrics but develops a cigar degeneracy. On $E(1,1)$ type closed manifolds (i.e., Sol geometries), the normalized Ricci flow exists for all positive time and presents a cigar degeneracy whereas the normalized cross curvature flow exists only for a finite time and there is a dimensional collapse with the sectional curvatures blowing up. The case of compact quotients of $SL(2,\mathbb{R})$ is the most difficult and perhaps the most interesting. The normalized Ricci flow exists for all time and presents a pancake type degeneracy. For the normalized negative cross curvature flow, two different types of behavior are possible. For metrics with a specific symmetry, the flow exists for all time and develops a pancake degeneracy. For generic (homogeneous) metrics, the flow exists for a finite time, there is a dimensional collapse and the curvatures blow up (to plus and minus infinity).

In the rest of this introduction, we quickly review the necessary material on locally homogenous 3-manifolds as well as the definition of the cross curvature flow. Sections 2 to 6 are devoted to the different examples: Heisenberg, $E(1,1)$ (i.e., Sol), SU(2), $SL(2,\mathbb{R})$ and $E(2)$.

1.2. **The cross curvature tensor on 3-manifolds.** On a 3-dimensional Riemannian manifold (M,g), let Rc be the Ricci tensor and R be the scalar curvature. The Einstein tensor is defined by $E = Rc - \frac{1}{2}Rg$, and its local components are $E_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$. Raising the indices, define $P^{ij} = g^{ik}g^{jl}R_{kl} - \frac{1}{2}Rg^{ij}$, where g^{ij} is the inverse of g_{ij}. Let V_{ij} be the inverse of P^{ij} (if it exists). The cross curvature tensor is

$$h_{ij} = \left(\frac{\det P^{kl}}{\det g^{kl}}\right) V_{ij}.$$

This definition is taken from [Chow and Hamilton 2004].

Assume that computation are done in an orthonormal frame where the Ricci tensor is diagonal. Then the cross curvature tensor is diagonal. If the principal sectional curvatures are k_1, k_2, k_3 ($k_i = K_{jkjk}$, circularly) so that $R_{ii} = k_j + k_l$ (circularly in i, j, l), then

$$h_{ii} = k_j k_l. \tag{1.1}$$

Notice that this definition works even when some of the sectional curvatures vanish (this was also addressed in [Ma and Chen 2006]).

The following lemma is proved in [Chow and Hamilton 2004] using the contracted second Bianchi identity,

Lemma 1. *(Chow and Hamilton [2004]). The cross curvature tensor h satisfies the following identities,*

$$\nabla_i P^{ij} = 0 \quad \textit{and} \quad (h^{-1})^{ij} \nabla_i h_{jk} = \frac{1}{2} \nabla_k h_{ij}.$$

Moreover, if the sectional curvatures are either positive or negative, then the identity map $id : (M, h) \to (M, g)$ is a harmonic map.

Recall that when the Ricci curvature tensor is positive (negative), the identity map $id : (M, g) \to (M, \pm Rc)$ is a harmonic map.

1.3. **The cross curvature flows.** Chow and Hamilton [2004] define the cross curvature flow on 3-manifolds having either positive sectional curvature or negative sectional curvature. The local existence of the flow, under any one of these two circumstances, was proved by Buckland [2006] . More precisely, if $\epsilon = \pm 1$ is the sectional curvature sign of the metric g_0, the cross curvature flow starting from g_0 is the solution of

$$\begin{cases} \frac{\partial}{\partial t} g = -2\epsilon h, \\ g(0) = g_0. \end{cases}$$

For the purpose of this paper, it should be noticed that locally homogeneous manifolds seldom have sectional curvatures that are all of the same sign. In dimension 3, positive sectional curvature is only possible on locally homogeneous manifolds covered by the sphere SU(2). Negative sectional curvature occurs only on hyperbolic 3-manifolds. All other locally homogeneous closed Riemannian 3-manifold are either flat or have some positive sectional curvatures [Milnor 1976, Theorem 1.6]. Thus the definition above is not really practical for our purpose. In fact, at least in the case of locally homogeneous 3-manifolds, it seems very natural to investigate both the positive and the negative cross curvature flows where the positive cross curvature flow is defined by

$$\begin{cases} \frac{\partial}{\partial t} g = 2h, \\ g(0) = g_0. \end{cases} \tag{+XCF}$$

and the negative cross curvature flow is defined by

$$\begin{cases} \frac{\partial}{\partial t} g = -2h, \\ g(0) = g_0. \end{cases} \tag{$-$XCF}$$

In this paper we consider the negative cross curvature flow ($-$XCF).

As in the Ricci flow, we can also consider the normalized cross curvature flow (NXCF) on closed 3-manifolds. It preserves the volume of closed 3-manifolds and is given by

$$\text{(NXCF)} \qquad \frac{\partial}{\partial t} g_{ij} = -2h_{ij} + \frac{2}{3}\bar{h} g_{ij},$$

where $\bar{h} = \int_{M^3} g^{ij} h_{ij} du / \int_{M^3} du$. As for the Ricci flow, the flows (-XCF) and (NXCF) only differ by a change of scale in space and a re-parametrization of time ($\tilde{g}(\tilde{t}) = \psi(t) g(t)$, $\tilde{t} = \int \psi^2(t)$).

1.4. **Locally homogeneous 3-manifolds.** Following Isenberg and Jackson [1992] (to which we refer for details concerning the following discussion), we take the view point that our original interest is in closed Riemannian 3-manifolds that are locally homogeneous. By a result of Singer [1960] , the universal cover of a locally homogeneous manifold is homogeneous, that is, its isometry group acts transitively. Now, since the cross curvature flow (just as the Ricci flow) commutes with the projection map from the universal cover, we can as well study the flow on the (often non-compact) universal cover.

In dimension 3 there are 9 possibilities for the universal cover, which can be labelled by the minimal isometry group that acts transitively:

(a) $H(3)$ ($H(n)$ denotes the isometry group of hyperbolic n-space); $\mathrm{SO}(3) \times \mathbb{R}$; $H(2) \times \mathbb{R}$;

(b) $\mathbb{R}^3$; SU(2); SL$(2, \mathbb{R})$; Heisenberg; $E(1,1) = $ Sol (the group of isometry of plane with a flat Lorentz metric); $E(2)$ (the group of isometries of the Euclidean plane).

The crucial difference between cases (a) and (b) above is that, in case (b), the universal cover of the corresponding closed 3-manifold is (essentially) the minimal transitive group of isometries itself (with the caveat that both SL$(2, \mathbb{R})$ and $E(2)$ should be replaced by their universal cover) whereas in case (a) this minimal group is of higher dimension. The cases (a) and (b) are studied separately. The case (b) is called the Bianchi case in [Isenberg and Jackson 1992]. It corresponds exactly to the classification of 3-dimensional simply connected unimodular Lie groups (nonunimodular Lie groups cannot cover a closed manifold).

1.5. **Real 3-dimensional unimodular Lie algebra.** The basis for our study is Milnor's description in [Milnor 1976, Section 4] of all three dimensional real Lie algebras equipped with an Euclidean structure (i.e., a left-invariant metric g_0 on the Lie group). Remember that the data here is the Lie algebra with a fixed Euclidean structure (and, in fact, a fixed orientation). The crucial result is as follows. Assume that $\mathfrak{g}$ is a 3-dimensional real unimodular Lie algebra equipped with an oriented Euclidean structure. Then there exists a (positively oriented) orthonormal basis (e_1, e_2, e_3) and reals $\lambda_1, \lambda_2, \lambda_3$ such that the bracket operation of the Lie algebra has the form

$$[e_i, e_j] = \lambda_k e_k \quad \text{(circularly in } i, j, k).$$

Milnor shows that such a basis diagonalizes the Ricci tensor and thus also the cross curvature tensor. If $f_i = a_j a_k e_i$ with nonzero $a_i, a_j, a_k \in \mathbb{R}$, then $[f_i, f_j] = \lambda_k a_k^2 f_k$ (circularly in i, j, k). Using the choice of orientation, we may assume that at most one of the λ_i is negative and then, the Lie algebra structure is entirely determined by the signs (in $\{-1, 0, +1\}$) of $\lambda_1, \lambda_2, \lambda_3$ as follows:

$+$	$+$	$+$	SU(2)
$+$	$+$	$-$	$\mathrm{SL}(2, \mathbb{R})$
$+$	$+$	0	$E(2)$ (Euclidean motions in $2D$)
$+$	0	$-$	$E(1, 1)$ (also called Sol)
$+$	0	0	Heisenberg group
0	0	0	$\mathbb{R}^3$

In each case, let $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in \{-1, 0, +1\}^3$ be the corresponding choice of signs. Then, given ϵ and an Euclidean metric g_0 on the corresponding Lie algebra, we can choose a basis f_1, f_2, f_3 (with f_i collinear to e_i above) such that

$$[f_i, f_j] = 2\epsilon_k f_k \quad \text{(circularly in } i, j, k). \tag{1.2}$$

As mentioned above, the metric, the Ricci tensor and the cross curvature tensor are diagonalized in this basis and this property is obviously maintained throughout either the Ricci flow or cross curvature flow. We call $(f_i)_1^3$ a Milnor frame for g_0. If we let $(f^i)_1^3$ be the dual frame of $(f_i)_1^3$, the metric g_0 is diagonalized by this frame and has the form

$$g_0 = A_0 f^1 \otimes f^1 + B_0 f^2 \otimes f^2 + C_0 f^3 \otimes f^3. \tag{1.3}$$

Assuming existence of the flow $g(t)$ starting from g_0, under either the Ricci flow or the cross curvature flow (positive or negative), the original frame $(f_i)_1^3$ stays a Milnor frame for $g(t)$ along the flow. Thus, $g(t)$ has the form

$$g(t) = A(t) f^1 \otimes f^1 + B(t) f^2 \otimes f^2 + C(t) f^3 \otimes f^3. \tag{1.4}$$

It follows that these flows reduce to ODEs in (A, B, C). Given a flow, the explicit form of the ODE depends on the underlying Lie algebra structure. With the help of the curvature computations done by Milnor in [1976] , one can find the explicit form of the equations for each Lie algebra structure. The Ricci flow case was treated in [Isenberg and Jackson 1992]. The case of the negative cross curvature flow is treated below.

1.6. **The trivial cases.** The three non-Bianchi cases and the flat case $\mathbb{R}^3$ all lead to essentially trivial behaviors. For $\mathbb{R}^3$, this is obvious.

In the hyperbolic case $H(3)$, the only homogeneous metrics are constant multiple of the standard hyperbolic metric. They all have constant negative curvature. The cross curvature tensor is a multiple of the identity. So each metric is a fixed point under the NXCF in this case.

In the case of $\mathrm{SO}(3) \times \mathbb{R}$, the homogeneous metrics must have a product form corresponding to a metric on $\mathbb{R}$ and a round metric on the 2 sphere. In a proper

frame, two of the principal sectional curvatures vanish and thus $h = 0$. The cross curvature flow is trivial.

Finally, for $H(2) \times \mathbb{R}$, the metrics again have a product form so that two of the three sectional curvatures vanish and $h = 0$. The cross curvature flow is trivial.

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2. The negative XCF on the Heisenberg group (Nil geometries)

Given a left-invariant metric g_0 on the Heisenberg group, fix a Milnor frame $\{f_i\}_1^3$ such that

$$[f_2, f_3] = 2f_1, \quad [f_3, f_1] = 0, \quad [f_1, f_2] = 0$$

and (1.3)-(1.4) hold. Using [Milnor 1976], the sectional curvatures are:

$$K(f_2 \wedge f_3) = -\frac{3A}{BC}, \quad K(f_3 \wedge f_1) = \frac{A}{BC}, \quad K(f_1 \wedge f_2) = \frac{A}{BC}.$$

The scalar curvature is $R = -2A/(BC)$. The computation of the cross curvature tensor easily follows by (1.1). In the frame $(f_i)_1^3$ and its dual frame $(f^i)_1^3$, the cross curvature tensor is given by

$$h = \frac{A^3}{B^2C^2} f^1 \otimes f^1 - 3\frac{A^2}{BC^2} f^2 \otimes f^2 - 3\frac{A^2}{B^2C} f^3 \otimes f^3.$$

Hence, the negative cross curvature flow ($-$XCF) reduces to the ODE system

$$\frac{dA}{dt} = -\frac{2A^3}{B^2C^2}, \quad \frac{dB}{dt} = \frac{6A^2}{BC^2}, \quad \frac{dC}{dt} = \frac{6A^2}{B^2C}.$$

Observe that

$$\frac{dA}{Adt} = -2\frac{A^2}{B^2C^2} = -\frac{1}{3}\frac{dB}{Bdt} = -\frac{1}{3}\frac{dC}{Cdt}.$$

Hence B/C, A^3B and A^3C stay constant under the flow and

$$\frac{dA}{dt} = -2\frac{A^3}{B^2C^2} = -2A^3 \cdot \frac{A^6}{A_0^6B_0^2} \cdot \frac{A^6}{A_0^6C_0^2} = -\frac{2}{A_0^{12}B_0^2C_0^2}A^{15}.$$

As $R_0 = -2A_0/(B_0C_0)$, we arrive at $A(t) = A_0(1 + 7R_0^2t)^{-\frac{1}{14}}$, $B(t) = B_0(1 + 7R_0^2t)^{\frac{3}{14}}$, and $C(t) = C_0(1 + 7R_0^2t)^{\frac{3}{14}}$. This shows that the solution of the flow exists for all time $t \geq 0$. The sectional curvatures are

$$K(f_2 \wedge f_3) = \frac{3R_0}{2}(1 + 7R_0^2t)^{-\frac{1}{2}} \text{ and } K(f_1 \wedge f_2) = K(f_3 \wedge f_1) = -\frac{R_0}{2}(1 + 7R_0^2t)^{-\frac{1}{2}}.$$

Hence we have the following result.

Theorem 1. *On the Heisenberg group, for any initial data A_0, B_0, $C_0 > 0$, the solution of the negative (XCF) on $[0, \infty)$ is given by*

$$A(t) = A_0(1 + 7R_0^2t)^{-\frac{1}{14}}, \quad B(t) = B_0(1 + 7R_0^2t)^{\frac{3}{14}} \text{ and } C(t) = C_0(1 + 7R_0^2t)^{\frac{3}{14}},$$

where $R_0 = -2A_0/(B_0C_0)$. The sectional curvatures decay as $t^{-1/2}$.

A closed manifold is a Nilmanifold if it is the quotient of a nilpotent Lie group by a discrete subgroup. A closed Riemannian manifold (M, g) is ϵ-flat ($\epsilon > 0$ fixed) if it admits a metric such that all sectional curvatures are bounded above in absolute value by ϵd^{-2} where d is the diameter of (M, g). A manifold is almost flat if it admits ϵ-flat metrics for all small $\epsilon > 0$. By a Theorem of M. Gromov [1978] (see also [Buser and Karcher 1981]), in any dimension, a manifolds is almost flat if and only if it is covered by a Nilmanifold.

The closed locally homogeneous 3-manifolds associated to the Heisenberg group are Nilmanifolds and thus are almost flat. Let $d(t)$ be the diameter of such a manifold under the negative XCF $g(t)$ considered above. Obviously, $d(t)^2$ is of order $t^{3/14}$ and the sectional curvatures are bounded in absolute value by a constant times $t^{-1/2}$. This shows that, as t tends to infinity, the negative XCF yields ϵ-flat metric (with $\epsilon(t)$ of order $t^{-2/7}$). The normalized flow (NXCF), has a similar behavior with a slightly different numerology.

3. The negative XCF on Sol geometry (E(1,1))

Given a left-invariant metric g_0 on $E(1, 1)$, fix a Milnor frame $\{f_i\}_1^3$ such that

$$[f_2, f_3] = 2f_1, \quad [f_3, f_1] = 0, \quad [f_1, f_2] = -2f_3.$$

The sectional curvatures are:

$$K(f_2 \wedge f_3) = \frac{(A - C)^2 - 4A^2}{ABC},$$

$$K(f_3 \wedge f_1) = \frac{(A + C)^2}{ABC},$$

$$K(f_1 \wedge f_2) = \frac{(A - C)^2 - 4C^2}{ABC}.$$

In the frame $\{f_1, f_2, f_3\}$, we have

$$(h_{ij}) = \begin{pmatrix} -\frac{A(A+C)^3(3C-A)}{(ABC)^2} & & \\ & \frac{B(3A-C)(3C-A)(A+C)^2}{(ABC)^2} & \\ & & -\frac{C(A+C)^3(3A-C)}{(ABC)^2} \end{pmatrix},$$

and the negative cross curvature flow equations are

$$\left\{ \begin{aligned} \frac{dA}{dt} =& 2\frac{A(A+C)^3(3C-A)}{(ABC)^2}, \\ \frac{dB}{dt} =& -2\frac{B(3A-C)(3C-A)(A+C)^2}{(ABC)^2}, \\ \frac{dC}{dt} =& 2\frac{C(A+C)^3(3A-C)}{(ABC)^2}. \end{aligned} \right. \tag{3.1}$$

If $A = C$ at $t = 0$, then $A(t) \equiv C(t)$ as long as the solution exists. Moreover,

$$\frac{dB}{dt} = -2\frac{4A^2 \cdot 4A^2}{A^2 A^2 B} = -\frac{32}{B},$$

so $B^2 = B_0^2 - 64t$, that is, $B = \sqrt{B_0^2 - 64t}$. Also, we have

$$\frac{d \ln A}{dt} = \frac{32}{B_0^2 - 64t}.$$

Hence

$$A(t) = C(t) = \frac{A_0 B_0}{\sqrt{B_0^2 - 64t}}.$$

For the case $A_0 \neq C_0$, we may assume without loss of generality that $A_0 > C_0$. Then we immediately have that C is increasing. Observing that

$$\begin{aligned}
\frac{d(A-C)}{dt} &= -2\frac{(A+C)^4}{(ABC)^2}(A-C),\\
\frac{d\ln(A/C)}{dt} &= -8\frac{(A+C)^3}{(ABC)^2}(A-C),\\
\frac{d(A-3C)}{dt} &= -2\frac{(A+C)^3}{(ABC)^2}(A^2+6AC-3C^2),
\end{aligned}$$

we find that $A > C$ and $A - C$, A/C and $A - 3C$ are decreasing.

Let us further assume that $3C_0 > A_0$. Then we have $1 < A/C < A_0/C_0 < 3$ and

$$B\frac{dB}{dt} = -2\frac{(A+C)^2}{(AC)^2}(3A-C)(3C-A) \in (-128, -E_0),$$

where $E_0 := 16(3C_0 - A_0)/A_0 > 0$. Therefore there exists $T_0 \in (0, \infty)$ such that $B(T_0) = 0$. Furthermore, when $t \in [0, T_0)$ $2E_0(T_0 - t) < B^2 < 256(T_0 - t)$. Hence

$$\frac{1}{C}\frac{dC}{dt} = 2\frac{(A+C)^3}{(ABC)^2}(3A-C) > \frac{16}{B^2} > \frac{1}{16(T_0-t)},$$

which implies that C and A go to ∞ as $t \to T_0^-$. It follows that, as $t \to T_0^-$,

$$B \sim \sqrt{64(T_0-t)}, \quad A, C \sim \frac{E_1}{\sqrt{T_0-t}}, \quad A - C \sim E_2\sqrt{T_0-t},$$

where E_1, E_2 are positive constants.

If $3C_0 \leq A_0$ then we claim that there exists $t_1 \geq 0$, such that $3C(t_1) > A(t_1)$ (and thus $3C(t) > A(t)$ for all $t > t_1$, as long as the solution exists since $A - 3C$ is decreasing). Suppose on the contrary that $3C \leq A$ as long as the solution exists, then we have that B is increasing, A is decreasing and

$$\begin{aligned}
B\frac{dB}{dt} &= 2\frac{(A+C)^2}{(AC)^2}(3A-C)(A-3C) < 6\frac{(A+C)^2}{C^2}\\
&= 6\left(1 + A/C\right)^2 < 6\left(1 + A_0/C_0\right)^2 := E_3.
\end{aligned}$$

Therefore the solution exists for all $t \in [0,\infty)$ and $B^2 < 2E_3 t + B_0^2$. Furthermore,

$$\begin{aligned}\frac{d(A-3C)}{dt} &= -2\frac{(A+C)^3}{(ABC)^2}(A^2+6AC-3C^2) < -\frac{16C^3\cdot 4C^2}{(ABC)^2} \\ &= -\frac{64C^3}{A^2B^2} < -\frac{64C_0^3}{A_0^2}\cdot\frac{1}{2E_3t+B_0^2}.\end{aligned}$$

Integrating the above inequality from 0 to ∞ yields a contradiction. Hence, we have the following theorem.

Theorem 2. *On $E(1,1)$, for any initial data A_0, B_0, $C_0 > 0$, there exists a time $T_0 > 0$, such that the solution of the negative cross curvature flow exists for all $0 \le t < T_0$. Moreover, as $t \to T_0^-$,*

$$B \sim \sqrt{64(T_0-t)}, \quad A, C \sim \frac{E_1}{\sqrt{T_0-t}}, \quad A-C \sim E_2\sqrt{T_0-t},$$

where E_1 and E_2 are constants. The sectional curvatures approach to $\pm\infty$ at rate of $(T_0-t)^{-1/2}$ as $t \to T_0$.

Remark. Under the normalized flow, the solution also only exists up to a finite time T_1, and

$$B \sim E_1'(T_1-t), \quad A, C \sim \frac{E_2'}{\sqrt{T_1-t}}.$$

The sectional curvatures approach to $\pm\infty$ at rate of $(T_1-t)^{-1/2}$ as $t \to T_1$. The diameter $d(t)$ increase like $(T_1-t)^{-1/4}$, so the absolute values of the sectional curvature are not $o(d(t)^{-2})$ (compare with the case of the Nil geometry). Recall that the solution to the normalized Ricci flow exists for all time and approaches a cigar degeneracy (see [Isenberg and Jackson 1992]), i.e., two directions shrink to zero while the other one expands to ∞, and the sectional curvatures decay at rate of t^{-1}. The Ricci flow and cross curvature flows behave quite differently in this case.

4. The negative XCF on $SU(2)$

Given a left-invariant metric g_0 on $SU(2)$, fix a Milnor frame such that $[f_i, f_j] = 2f_k$ circularly. We have $K(f_2\wedge f_3) = \frac{(B-C)^2}{ABC} - \frac{3A}{BC} + \frac{2}{B} + \frac{2}{C}$ and the other sectional curvatures are obtained by circular permutation. The cross curvature tensor is diagonal under the associated orthogonal frame $\{f_i\}_1^3$ with $h_{11} = (ABC)^{-2}AYZ$ and the other entries obtained again by circular permutation with

$$\begin{aligned}X =& 3A^2 - (B-C)^2 - 2AB - 2AC, \\ Y =& 3B^2 - (A-C)^2 - 2AB - 2BC, \\ Z =& 3C^2 - (A-B)^2 - 2BC - 2AC.\end{aligned}$$

Therefore, under ($-$XCF), A, B, C satisfy the following equations

$$\begin{cases} \dfrac{dA}{dt} &= -2\dfrac{AYZ}{(ABC)^2}, \\ \dfrac{dB}{dt} &= -2\dfrac{BZX}{(ABC)^2}, \\ \dfrac{dC}{dt} &= -2\dfrac{CXY}{(ABC)^2}. \end{cases} \tag{4.1}$$

Without loss of generality we may assume that $A_0 \geq B_0 \geq C_0$. Then we know that $A \geq B \geq C$ as long as a solution exists. Observing that

$$\begin{aligned} Y &= (B-A)(A+B+2B-2C) - C^2 \leq -C^2 < 0, \\ Z &= -(A-B)^2 + 3C^2 - 2AC - 2BC \leq -C^2 < 0, \end{aligned}$$

we have

$$\begin{aligned} \frac{d(A-B)}{dt} &= \frac{2Z}{(ABC)^2}(A-B)(A^2 + A(6B-2C) + (B-C)^2) \leq 0, \\ \frac{d(A-C)}{dt} &= \frac{2Y}{(ABC)^2}(A-C)((A-B)^2 + 6AC - 2BC + C^2) \leq 0, \\ \frac{d\ln(A/B)}{dt} &= \frac{8Z}{(ABC)^2}(A-B)(A+B-C) \leq 0, \\ \frac{d\ln(A/C)}{dt} &= \frac{8Y}{(ABC)^2}(A-C)(A+C-B) \leq 0. \end{aligned}$$

It follows that A, $A-B$, $A-C$, $\ln(A/B)$ and $\ln(A/C)$ are decreasing. Furthermore,

$$-\frac{dA}{dt} = \frac{2AYZ}{(ABC)^2} \geq \frac{2AC^4}{(A^2C)^2} = \frac{2C^2}{A^3} \geq \frac{C_0^2}{A_0^2}\frac{2}{A},$$

which implies $\frac{d}{dt}A^2 \leq -4C_0^2A_0^{-2}$. Therefore there exists $T_0 \in (0, \infty)$, such that $A(T_0) = B(T_0) = C(T_0) = 0$. On the other hand

$$-\frac{dA}{dt} = \frac{2AYZ}{(ABC)^2} \leq \frac{2A}{(AC^2)^2}(3A^2)(4A^2) = \frac{24A^3}{C^4} \leq \frac{24A_0^4}{C_0^4}\frac{1}{A}.$$

It follows that on $[0, T_0)$,

$$\sqrt{48(T_0 - t)}\frac{A_0^2}{C_0^2} \geq A(t) \geq 2\sqrt{T_0 - t}\frac{C_0}{A_0}. \tag{4.2}$$

Since A/C is decreasing and bounded below by 1, we may assume that $\lim_{T_0^-} A/C = p$. We claim that $p = 1$. Suppose instead that $p > 1$. Then we have

$$-\frac{d\ln(A/C)}{dt} = \frac{-4Y}{(ABC)^2}(A-C)(A+C-B) \geq \frac{4C^2(A-C)C}{(A^2C)^2} \geq (1-p^{-1})\frac{C_0}{A_0}\frac{4}{A^2}.$$

Integrating from 0 to T_0 and using (4.2), we get a contradiction. Therefore, $\lim_{T_0^-} A/C = 1$. It follows easily from (4.1) that as $t \to T_0^-$, $A, B, C \sim 2\sqrt{T_0 - t}$.

Theorem 3. *For any choice of initial data A_0, B_0, $C_0 > 0$, there exists a time $T_0 > 0$, such that the solution of the cross curvature flow on $SU(2)$ exists for all $0 \le t < T_0$. Moreover, as $t \to T_0^-$,*

$$A, B, C \sim 2\sqrt{T_0 - t}.$$

Remark. If we consider the normalized XCF, we find that solutions of (NXCF) exists for all time. As $t \to \infty$, A, B and C approach a constant and the sectional curvatures also become constant, so the metric becomes round. Since the sectional curvatures become positive for t large enough, according to Chow and Hamilton, the negative XCF is the more natural choice in this case.

5. The negative XCF on $SL(2, \mathbb{R})$

Given a left-invariant metric g_0 on $SL(2, \mathbb{R})$, fix a Milnor frame $\{f_i\}_1^3$ such that

$$[f_2, f_3] = -2f_1, \quad [f_3, f_1] = 2f_2, \quad [f_1, f_2] = 2f_3.$$

The sectional curvatures are

$$\begin{aligned} K(f_2 \wedge f_3) &= \frac{1}{ABC}(-3A^2 + B^2 + C^2 - 2BC - 2AC - 2AB), \\ K(f_3 \wedge f_1) &= \frac{1}{ABC}(-3B^2 + A^2 + C^2 + 2BC + 2AC - 2AB), \\ K(f_1 \wedge f_2) &= \frac{1}{ABC}(-3C^2 + A^2 + B^2 + 2BC - 2AC + 2AB), \end{aligned}$$

and the cross curvature tensor under the associate frame $(f_i)_1^3$ is

$$(h_{ij}) = \frac{1}{(ABC)^2} \begin{pmatrix} AF_2F_3 & & \\ & BF_3F_1 & \\ & & CF_1F_2 \end{pmatrix},$$

where

$$\begin{aligned} F_1 &= -\,3A^2 + B^2 + C^2 - 2BC - 2AC - 2AB, \\ F_2 &= -\,3B^2 + A^2 + C^2 + 2BC + 2AC - 2AB, \\ F_3 &= -\,3C^2 + A^2 + B^2 + 2BC - 2AC + 2AB. \end{aligned}$$

Therefore, under the negative XCF, A, B, C satisfy the following equations

$$\begin{cases} \dfrac{dA}{dt} = -\dfrac{2AF_2F_3}{(ABC)^2}, \\ \dfrac{dB}{dt} = -\dfrac{2BF_3F_1}{(ABC)^2}, \\ \dfrac{dC}{dt} = -\dfrac{2CF_1F_2}{(ABC)^2}. \end{cases} \tag{5.1}$$

If $B_0 = C_0$, then $B = C$ as long as a solution exists and A, B satisfy

$$\frac{dA}{dt} = -\frac{2A^3}{B^4}, \quad \frac{dB}{dt} = 2\frac{3A^2 + 4AB}{B^3}.$$

Then A is decreasing and B is increasing. We have

$$\frac{dA^{-2}}{dt} = -2A^{-3}\frac{dA}{dt} \leq 4B_0^{-4},$$
$$\frac{dB^3}{dt} \leq 6\frac{3A_0^2 + 4A_0 B}{B} \leq C_1,$$

where $C_1 = 24A_0 + 18A_0^2 B_0^{-1}$ is a constant. Integrating from 0 to t, we obtain that

$$A \geq (4B_0^{-4}t + A_0^{-2})^{-\frac{1}{2}},$$
$$B \leq (C_1 t + B_0^3)^{\frac{1}{3}}.$$

It follows that a solution exists on $[0, \infty)$.

$$\frac{d(4A^{-1} + B^{-1})}{dt} = -4A^{-2}\frac{dA}{dt} - B^{-2}\frac{dB}{dt} = -6\frac{A^2}{B^5},$$
$$\frac{d(A^9 B^3)}{dt} = 9A^8 B^3\frac{dA}{dt} + 3A^9 B^2\frac{dB}{dt} = 24A^{10}. \tag{5.2}$$

Hence $4A^{-1} + B^{-1}$ is decreasing, which implies $\lim_{t\to\infty} A := A_\infty > 0$. Integrating (5.2) we obtain that $A^9 B^3 \to \infty$ as $t \to \infty$. Hence $\lim_{t\to\infty} B = \infty$. It is not hard to show that as $t \to \infty$,

$$A \sim A_\infty + \frac{1}{8\sqrt[3]{3}} A_\infty^{\frac{5}{3}} t^{-\frac{1}{3}}, \qquad B \sim (24A_\infty t)^{\frac{1}{3}}.$$

For the case $B_0 \neq C_0$, we may assume without loss of generality that $B_0 > C_0$. Then $B > C$ as long as a solution exists. It follows that

$$F_3 = (B - C)(2A + B + 3C) + A^2 > A^2 > 0.$$

Let $a = AB^{-1}$ and $c = CB^{-1}$.

Lemma 2. *Suppose that at $t = 0$, a and c satisfy*

$$a < 1 - c + 2\sqrt{1 - c} \tag{5.3}$$

and

$$a > \frac{1}{3}(2\sqrt{1 - c + c^2} - 1 - c). \tag{5.4}$$

Then a and c satisfy (5.3) and (5.4) as long as a solution exists.

Proof. As

$$F_2 = (A - (B - C + 2\sqrt{(B - C)B}))(A - (B - C - 2\sqrt{(B - C)B})),$$
$$F_1 = (B - (A + C + 2\sqrt{(A + C)A}))(B - (A + C - 2\sqrt{(A + C)A})).$$

we see that (5.4) is equivalent to $F_1 < 0$ and (5.3) is equivalent to $F_2 < 0$. Since

$$\left.\frac{dA}{dt}\right|_{F_2=0} = 0, \quad \left.\frac{dB}{dt}\right|_{F_2=0} > 0 \quad \text{and} \quad \left.\frac{dC}{dt}\right|_{F_2=0} = 0$$

we obtain that

$$\left.\frac{dF_2}{dt}\right|_{F_2=0} < 0. \tag{5.5}$$

To prove that $F_2(t) < 0$ we argue by contradiction. Suppose t_0 is the first time such that $F_2(t_0) = 0$. Since $F_2(0) < 0$, we know that $F_2'(t_0) \geq 0$, which contradicts (5.5). Therefore $F_2(t) < 0$, which is equivalent to (5.3). Similarly $\left.\frac{dF_1}{dt}\right|_{F_1=0} < 0$ and $F_1(t) < 0$. This completes the proof of the lemma. $\square$

Lemma 3. *Suppose that at $t = 0$, a and c satisfy*

$$a \geq 1 - c + 2\sqrt{1-c} \tag{5.6}$$

Then eventually a and c will satisfy (5.3) and (5.4).

Proof. Suppose on the contrary that $a \geq 1 - c + 2\sqrt{1-c}$ always holds, then $F_2 > 0$ and thus $F_1 < 0$. Hence A is decreasing and B, C are increasing as long as a solution exists. Note that

$$\frac{d\ln(C/B)}{dt} = \frac{8(B-C)}{(ABC)^2}(A+B+C)F_1 < 0.$$

Therefore $c = C/B$ is decreasing, which implies that

$$A = aB \geq (1 - c + 2\sqrt{1-c})B \geq (1 - c_0 + 2\sqrt{1-c_0})B_0.$$

On the other hand

$$\frac{dB^2}{dt} = \frac{-4F_1}{A^2} \cdot \frac{F_3}{C^2}.$$

As

$$F_3/C^2 \geq 3((B/C) - 1) \geq 3((B_0/C_0) - 1)$$

and

$$-F_1 = F_2 + 2((A+B)^2 - C^2) \geq B^2 - C^2 \geq C_0^2((B_0/C_0)^2 - 1),$$

it follows that $(B^2)' > \eta$ for some positive constant η. If B stays finite for all t then the solution exists on $[0, \infty)$ and $\lim_{t\to\infty} B = \infty$. Therefore as $t \to \infty$

$$F_2 = -(B-C)(2A + C + 3B) + A^2 \leq -3BC(\frac{B}{C} - 1) + A^2 \to -\infty.$$

This contradicts $F_2 > 0$. If B goes to ∞ in finite time then $F_2 \to -\infty$ in finite time. We also get a contradiction. $\square$

Lemma 4. *Suppose that at $t = 0$, a and c satisfy*

$$a \leq \frac{1}{3}(2\sqrt{1 - c + c^2} - 1 - c). \tag{5.7}$$

Then eventually a and c will satisfy (5.3) and (5.4).

Proof. Suppose on the contrary that $a \le \frac{1}{3}(2\sqrt{1-c+c^2}-1-c)$ always holds. Then $F_1 > 0$ and thus $F_2 < 0$. It follows that A, C are increasing and B is decreasing as long as a solution exists. Since

$$-F_2 = F_1 + 2((A+B)^2 - C^2) \ge 2A^2 + 4AB \text{ and } F_3 \ge A^2,$$

we obtain that $A' \ge 4A^3(BC)^{-2} \ge 4A_0^3 B_0^{-4}$. If A stays finite for all t then the solution exists on $[0,\infty)$ and $\lim_{t\to\infty} A = \infty$. Therefore as $t \to \infty$

$$F_1 = (B-C)^2 - A(3A+2C+2B) \le (B_0 - C_0)^2 - 3A^2 \to -\infty.$$

This is a contradiction. If A goes to ∞ in finite time then $F_1 \to -\infty$ in finite time. We also get a contradiction. ☐

From the above three lemmas, we can assume without loss of generality that (5.3) and (5.4) hold at $t = 0$. Then $F_1 < 0$, $F_2 < 0$, A, B are increasing and C is decreasing as long as a solution exists.

Lemma 5. *Suppose that in addition we have $A_0 + C_0 \le B_0$, then there exists $T > 0$ such that*

$$A, B \sim E(T-t)^{-\frac{1}{2}}, \quad C \sim 8\sqrt{T-t} \qquad \text{as } t \to T^-, \tag{5.8}$$

where E is a positive constant.

Proof. We first claim that $A + C \le B$ holds for all t. In fact we have

$$\frac{d\ln(A/B)}{dt} = -\frac{8(A+B)}{(ABC)^2} F_3(A+C-B), \tag{5.9}$$

which implies $\frac{d(A/B)}{dt}\Big|_{A+C-B=0} = 0$. On the other hand $\frac{d(C/B)}{dt}\Big|_{A+C-B=0} < 0$. It follows that

$$\frac{d(A+C-B)}{dt}\Bigg|_{A+C-B=0} < 0,$$

which implies $A + C \le B$ as long as a solution exists. Therefore $a = AB^{-1}$ is increasing and $-F_2 \ge (A+B)^2 - C^2 \ge 2A(A+B+C)$. It follows that

$$\frac{d\ln A}{dt} = \frac{2(-F_2)F_3}{(ABC)^2} > \frac{4AB\cdot A^2}{(ABC)^2} = \frac{4A}{BC^2} \ge \frac{4A_0}{B_0 C_0^2}.$$

If the solution exists on $[0,\infty)$, integrating the above inequality from 0 to ∞ yields $\lim_{t\to\infty} A = \infty$. Since $A + C \le B$, we also obtain that $\lim_{t\to\infty} B = \infty$. Let $p := \lim_{t\to\infty} A/B$. Then p must be 1, otherwise integrating (5.9) from 0 to ∞ we get a contradiction. Since $\lim_{t\to\infty} A/B = 1$, as $t \to \infty$, $F_1, F_2 \sim -(A+B)^2$. It follows from

$$\frac{dC^2}{dt} = -4\frac{F_1 F_2}{A^2 B^2},$$

that C goes to 0 in finite time. This contradicts the assumption that a solution exists on $[0,\infty)$. Therefore any solution blows up in finite time. Suppose $[0, T)$ is the maximal time interval of a solution. Let $C_T = \lim_{t\to T^-} C$. If $C_T > 0$, then it follows easily from (5.1) that A, B stay bounded on $[0, T)$. Therefore we may extend

the solution beyond T, which contradicts the maximality of the time interval $[0, T)$. Hence $\lim_{t\to T^-} C = 0$. From

$$\frac{d(AC)}{dt} = -F_2 \frac{4AC}{(ABC)^2}(B^2 - (A+C)^2) \geq 0,$$

we obtain that $A \to \infty$ and $B \geq A + C \to \infty$ as $t \to T^-$. Let $p := \lim_{t\to T^-} AB^{-1}$. Since $A + C \leq B$, we have $p \leq 1$. If $p < 1$, then as $t \to T^-$,

$$\frac{d(A-B)}{dt} = \frac{2F_3}{(ABC)^2}(BF_1 - AF_2) = \frac{2F_3}{(ABC)^2}((A+B)^2(B-A-2C)+(B-A)C^2) > 0,$$

which is impossible since $A - B \sim -(1-p)B$. Therefore $\lim_{t\to T^-} AB^{-1} = 1$ and

$$\lim_{t\to T^-} A^{-2}F_1 = \lim_{t\to T^-} A^{-2}F_2 = -4 \text{ and } \lim_{t\to T^-} A^{-2}F_3 = 4.$$

Then (5.8) follows easily from (5.1). $\square$

The only case left now is that A, B, C satisfy $A_0 + C_0 > B_0$ and $F_1(0), F_2(0) < 0$. If there is a time T^*, such that $A(T^*) + C(T^*) = B(T^*)$, then from Lemma 5 we know the behavior of the solution. If on the other hand $A + C > B$ for all t, then

$$-F_1 \geq (A+B)^2 - C^2$$

and (5.9) implies that AB^{-1} is decreasing. It follows that

$$-F_2 = (B - C + 2\sqrt{B(B-C)} - A)(A - (B - C - 2\sqrt{B(B-C)})) \geq C_1 AB,$$

where C_1 is some constant depending only on A_0, B_0 and C_0. Using the above two inequalities and (5.1) we obtain that

$$\frac{dC^2}{dt} = -4\frac{F_1F_2}{(AB)^2} \leq -4\frac{C_1((A+B)^2 - C^2)}{AB} \leq -8C_1.$$

Therefore, there exist a finite $T > 0$ such that $C(T) = 0$. Form

$$\frac{d(BC)}{dt} = -F_1 \frac{4BC}{(ABC)^2}(A^2 - (B-C)^2) \geq 0,$$

we obtain that $B \to \infty$ and $A > B - C \to \infty$ as $t \to T^-$. Again let $p := \lim_{t\to T^-} AB^{-1}$. Since $A + C > B$, we have $p \geq 1$. If $p > 1$, then as $t \to T^-$,

$$\frac{d(A-B)}{dt} = \frac{2F_3}{(ABC)^2}(BF_1 - AF_2) = \frac{2F_3}{(ABC)^2}((A+B)^2(B-A-2C)+(B-A)C^2) < 0,$$

which is impossible since $A - B \sim (p-1)B$. Therefore $\lim_{t\to T^-} AB^{-1} = 1$,

$$\lim_{t\to T^-} A^{-2}F_1 = \lim_{t\to T^-} A^{-2}F_2 = -4 \text{ and } \lim_{t\to T^-} A^{-2}F_3 = 4$$

and we have (5.8). Hence we have the following theorem.

Theorem 4. *On $SL(2,R)$, for given initial data A_0, B_0, $C_0>0$, if $B_0=C_0$, then $B(t)=C(t)$, and the solution of the negative cross curvature flow exists for all $t\in[0,\infty)$. As functions of t, A is decreasing and $\lim_{t\to\infty}A=A_\infty>0$, whereas B,C are increasing and go to ∞. Moreover, as $t\to\infty$*

$$A\sim A_\infty+\frac{1}{8\sqrt[3]{3}}A_\infty^{\frac{5}{3}}t^{-\frac{1}{3}},\qquad B=C\sim(24A_\infty t)^{\frac{1}{3}},$$

the sectional curvatures all approach to 0 as $t\to\infty$ ($K(f_2\wedge f_3)\sim-\frac{4}{B}\sim-E_1t^{-1/3}$ and $K(f_3\wedge f_1)=K(f_1\wedge f_2)=\frac{A}{BC}\sim E_2t^{-2/3}$, where E_1 and E_2 are some constants).

If $B_0>C_0$, then there exists a time $T_0>0$, such that the solution of the cross curvature flow on $SL(2,\mathbf{R})$ exists for all $0\le t<T_0$. Moreover, as $t\to T_0^-$

$$A,B\sim E(T_0-t)^{-\frac{1}{2}},\quad C\sim 8\sqrt{T_0-t},$$

where E is some constant, and all sectional curvatures go to $\pm\infty$ at the rate of $(T_0-t)^{-1/2}$ as $t\to T_0$.

Remark. The asymptotic behavior of the solution depends on the initial data in this case, i.e., the condition $B=C$ does not characterize the typical geometry of general solution. If we consider the normalized cross curvature flow, then for the case of $B=C$, $A\sim E_1t^{-\frac{2}{5}}$, $B=C\sim E_2t^{\frac{1}{5}}$, where E_1 and E_2 are some constants, one sectional curvature decays at rate of $t^{-\frac{1}{5}}$ and the other two sectional curvatures decay at rate of $t^{-\frac{4}{5}}$, we have a pancake degeneracy. For the case of $B\neq C$ under (NXCF), $A,B\sim E_1(T_1-t)^{-\frac{1}{4}}$, $C\sim E_2\sqrt{T_1-t}$, all sectional curvatures go to $\pm\infty$ at the rate of $(T_1-t)^{-1/2}$ as $t\to T_1$, where T_1 is the maximal existence time for the solution. Recall that the solution of the Ricci flow in this case exists for all time and develops a pancake degeneracy.

6. The negative XCF on $E(2)$

Given a left-invariant metric g_0, fix a Milnor frame $\{f_i\}_1^3$ such that

$$[f_2,f_3]=2f_1,\ \ [f_3,f_1]=2f_2,\ \ [f_1,f_2]=0.$$

The sectional curvatures are

$$K(f_2\wedge f_3)=\frac{1}{ABC}(B-A)(B+3A),$$

$$K(f_3\wedge f_1)=\frac{1}{ABC}(A-B)(A+3B),$$

$$K(f_1\wedge f_2)=\frac{1}{ABC}(A-B)^2,$$

and the cross curvature tensor in the frame $\{f_i\}_1^3$ is

$$(h_{ij})=\frac{1}{(ABC)^2}\begin{pmatrix}AYZ&&\\&BZX&\\&&CXY\end{pmatrix},$$

where

$$X = (A - B)(3A + B),\ Y = (B - A)(3B + A) \text{ and } Z = -(A - B)^2.$$

Therefore, under the negative XCF, A, B, C satisfy the following equations

$$\begin{cases} \dfrac{dA}{dt} = -\dfrac{2A(3B + A)(A - B)^3}{(ABC)^2}, \\ \dfrac{dB}{dt} = -\dfrac{2B(3A + B)(B - A)^3}{(ABC)^2}, \\ \dfrac{dC}{dt} = \dfrac{2C(3A + B)(3B + A)(A - B)^2}{(ABC)^2}. \end{cases} \tag{6.1}$$

If $A_0 = B_0$, then the geometry stays flat at all time. Without loss of generality, we assume that $A_0 > B_0$. Then we have $B_0 \leq B(t) < A(t) \leq A_0$ as long as the solution exists. Since $C'(t) > 0$, $C(t)$ is increasing. It follows easily from (6.1) that a solution exists for all $t \in [0, \infty)$. We first claim that $\lim_{t\to\infty} C = \infty$. In fact, suppose that $\lim_{t\to\infty} C = C_\infty < \infty$, then

$$\frac{d(A - B)}{dt} = -2\frac{(A - B)^3}{(ABC)^2}(A^2 + 6AB + B^2)$$

implies $((A - B)^{-2})' \sim E_1$ as $t \to \infty$, where E_1 is some constant. It follows that $A - B \sim E_1^{-\frac{1}{2}} t^{-\frac{1}{2}}$. Then, from

$$\frac{dC^2}{dt} = 4\frac{(3A + B)(3B + A)}{(AB)^2}(A - B)^2,$$

we obtain that $C \to \infty$ as $t \to \infty$. This is a contradiction. Now,

$$\frac{d\ln((A - B)^2 C)}{dt} = 2\frac{d\ln(A - B)}{dt} + \frac{d\ln C}{dt} = \frac{2(A - B)^4(A + B)}{(ABC)^2} \leq E_2\frac{dB}{dt},$$

for some positive constant E_2. Therefore $(A - B)^2 C$ is increasing and approaches some finite number as $t \to \infty$. Hence

$$C^2\frac{dC}{dt} = 2\frac{(3A + B)(3B + A)}{(AB)^2}(A - B)^2 C \sim E_3,$$

as $t \to \infty$. It follows that $C \sim E_4 t^{\frac{1}{3}}$, $A - B \sim E_5 t^{-\frac{1}{6}}$ and

$$\frac{d(A + B)}{dt} = -2\frac{(A - B)^4(A + B)}{(ABC)^2} \sim E_6 t^{-\frac{4}{3}}.$$

Therefore we have the following theorem.

Theorem 5. *On $E(2)$, for any initial data A_0, B_0, $C_0 > 0$, if $A_0 = B_0$, then the solution of $(-XCF)$ exists for all time, $A(t) = B(t) = A_0$ and $C(t) = C_0$ for all time t (the geometry stays flat).*

If $A_0 > B_0$, then the solution exists for all $t \in [0, \infty)$ and, as $t \to \infty$

$$A \sim E_1 + E_2 t^{-\frac{1}{6}}, B \sim E_1 - E_2 t^{-\frac{1}{6}} \text{ and } C \sim (8E_2/E_1)\sqrt{6}t^{\frac{1}{3}},$$

where E_1 and E_2 are positive constants. Two of the sectional curvatures decay like $t^{-1/2}$, while the other one decays like $t^{-2/3}$.

Remark. Under the Ricci flow, the geometry converges to a flat metric. If we consider the solution to the normalized cross curvature flow, in the case that $A_0 = B_0$, we still have flat metric. For the case of $A_0 \neq B_0$, we have

$$A \sim E_1 t^{-\frac{1}{7}}, B \sim E_1 t^{-\frac{1}{7}} \text{ and } C \sim E_2 t^{\frac{2}{7}},$$

while two of the sectional curvatures decay like $t^{-1/2}$, and the other one decays like $t^{-5/7}$, hence the solution of (NXCF) develops a cigar degeneracy, i.e., two directions shrink to zero, the other one expands without bound, while the sectional curvature dies off.

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